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Chaos in Economics[1]

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Abstract

Chaos theory has generated a lot of excitement and important results in physics and some other fields. Can we say the same as regards economics? This paper briefly surveys the large body of literature on chaos in economics, about which much has been written. More specifically, the attention will be devoted to discuss the extent to which the techniques to approach chaos in economics can address market anomalies, complexity and chaotic phenomena. The various ways to detect chaotic behavior in economics can be well exemplified by considering exchange rates. We consider a continuous time exchange rate model that allows for heterogeneity of the agents' beliefs, in order to explore non-linearities and possible chaotic behavior since we believe that the correct approach is to try to find the dynamic model (if any) underlying the data. The model is econometrically estimated with Euro/Dollar data and examined for the possible presence of chaotic motion. The results indicate that the possibility of chaotic dynamics in our model is rejected.

Keywords: chaos, economic dynamics, exchange rates, continuous time econometrics

JEL Classification: C610, C490, F310, F370

1. Introduction

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The interest of economists in chaos theory started in the 1980s, more than 20 years after the onset of this theory in physics, which is conventionally dated from 1963 when the meteorologist E. N. Lorenz published his paper on what became to be known as the Lorenz attractor (Lorenz, 1963). A few words are in order to explain the interest of economists in chaos. One of the reasons is the fact *chaos is apparently stochastic behavior generated by a dynamic deterministic system*.

By “apparently stochastic”, we mean a random path that at first sight cannot be distinguished from the path generated by a stochastic variable. To illustrate this, rather than using the traditional examples such as the Lorenz weather equations etc., we take from Brock et al. (1991) the simple example of a computer pseudo random number generator. The algorithm used by the computer is purely deterministic, but what comes out is a series of numbers that looks random, and that will fool any statistician in the sense that it passes all the standard tests of randomness. Actually, random numbers generated in this way are usually employed in statistical analysis.

A second feature that is often cited as typical of chaotic behavior of deterministic systems is the *impossibility of predicting the future values of the variable(s) concerned*. This might at first sight seem a contradiction - if we have a dynamic deterministic system, even if we cannot solve it analytically - we can simulate it numerically, hence we can compute the value(s) of the variable(s) for any future value of t. This is where another important feature of chaos comes in, that is *sensitive dependence on initial conditions*.

Sensitive dependence on initial conditions (henceforth *SDIC*), also called the “*butterflyeffect*[4]”, means that even very small differences in the initial conditions eventually give rise to widely different paths. Oppositely, in a “normal” deterministic system, all nearby paths starting very close to one another remain very close in the future. Hence, a sufficiently small measurement error in the initial conditions will not affect our deterministic forecasts.

On the contrary, in deterministic systems with *SDIC*, prediction of the future values of the variable(s) would be possible only if the initial conditions could be measured with infinite precision. This is certainly not the case.

[4]E.N. Lorenz introduced the terminology “butterfly effect” to denote *SDIC* in a talk given in 1972, for the idea that a butterfly flapping its wings in Brazil could cause a hurricane in Texas. The talk is reproduced as an appendix in E.N. Lorenz, 1993. The flapping wings represent a tiny change in the initial conditions of the earth atmospheric system, a change that causes large-scale alterations of events.

All this had already been noted by Poincaré (1908, p. 68), whose often-cited sentence runs as follows:

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. [...] It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon".

Poincaré's work was apparently unknown to the mathematician who is said to have claimed (in the ante-chaos era) that, given a mega computer and sufficient funds to collect data, weather could be forecast with accuracy. If the weather equations are chaotic, in the sense that they exhibit *SDIC* (and they do), no amount of funds and no supercomputer will ever yield the required infinite precision. It was in fact after the meteorologist E.N. Lorenz (1963) found *SDIC* in a system of three differential equations emerging in the theory of turbulence in fluids, that the mathematical study of chaos blossomed (Guckenheimer and Holmes, 1986, Chap. 2, Sect. 2.3).
A generally accepted mathematical definition of chaos does not yet exist. Some take *SDIC* as the hallmark of chaos: see, for example, Brock et al., 1991, p. 9; similarly Strogatz (1994, pp. 323-324) suggests the following working definition:

"*Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions*".

Aperiodic long-term behavior means that there are trajectories that do not settle down to fixed points, periodic orbits, or quasi-periodic[5] orbits as $t \to \infty$.

Others argue that chaotic dynamics depends on the existence of a strange attractor (Guckenheimer and Holmes, 1986), that Mandelbrot (1983) calls a fractal attractor (on these distinctions, which are not only terminological, see Rosser (2000, Sect. 2.4) and Mirowski(1990).

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[5]We have quasi-periodicity when every trajectory winds around endlessly on a torus, never intersecting itself and yet never quite closing (Strogatz, 1994, p. 276).

Other definitions also exist (Rosser, 2000, Sect. 2.3.2.2.5). In what follows, we shall accept Strogatz's definition and adopt a pragmatic approach.

The aim of the paper is not to survey the large body of literature concerning the development of methods to detect chaotic behavior, about which much has been written, but rather to review the results reached by their application in analyzing economic data. More specifically, we discuss the extent to which they can address the complexity, market anomalies, and stylized facts in economic markets. In addition, we will show how nonlinear heterogeneous agent models can characterize the dynamics of exchange rate markets, with fundamental nonlinear structures, subject to internal and external forces that may be sources of chaos.

2. Chaos in Economics

Chaos theory has generated a lot of excitement and important results in physics and some other fields. Can we say the same as regards economics? Economists are well known for drawing heavily from physics (Mirowski, 1990, 1992) but, in general, a successful theory in one field is not automatically a serious theory in another field[6]. The excitement does exist in economics, as shown by the numerous papers (collections are contained for example in Anderson et al. eds,1988; Grandmont ed., 1988;Barnett et al. eds., 1989, 1996; Bischi et al. eds., 2013; Benhabib ed., 1992; Day and Chen eds., 1993; Creedy and Martin eds., 1994;Leydesdorff and Van den Besselaar eds., 1994, Rosser and Kremer eds., 2004; Kyrtsou and Palivos eds., 2006;Faggini 2008, 2009) and the several books (see, e.g., Brock et al., 1991; Chiarella, 1990; Day, 1994; H.-W. Lorenz, 1993; Medio, 1992; Medio and Lines, 2001; Peters, 1991, 1994; Puu, 2003; Rosser, 2000, 2009, 2011; Lines ed., 2005; Zhang, 2005, 2006) on chaotic economic dynamics.

These show that chaos theory in economics is not a fad. They also show that, from the theoretical point of view, plausible economic models can be built, and old economic models can be revisited, in which chaotic behavior is present, although sometimes the assumptions may look a bit *ad hoc* (on this point see for example Sordi, 1993;Nusse and Hommes, 1990).

[6] In our opinion, the application of chaos theory has been only a mechanical transfer that has not taken into account the specific features of economic systems.

We know that *periodic* oscillations can arise in standard non-linear models, but before the advent of chaos to explain the aperiodicity often observed in actual economic variables it was necessary to rely on an unexplained *exogenous* random variable. Chaos, on the contrary, gives us an *endogenous* explanation of erraticity. As Goodwin (1991, p. 425) aptly put it, "Poincare generalized an equilibrium point to an equilibrium motion; a chaotic attractor generalizes the motion to a bounded equilibrium region towards which all motions tend, or within which all motions remain; the conception of equilibrium is more or less lost since all degrees of a periodic, or erratic fluctuations can occur within the region. *The special relevance of this to economics is that it offers not* one *but* two *types of explanation of the pervasive irregularity of economic time-series - an endogenous one in addition to the conventional exogenous shock*" (emphasis added).

Let us now come to three other major theoretical implications of chaos.

The first is that the rational expectations hypothesis is untenable in the face of chaos (Chiarella, 1990, pp. 124-125; Kelsey, 1988, pp. 682-683; Medio, 1993, pp. 17-18). It should be stressed that this is *not* a criticism to rational expectations of the type: it is practically impossible that all economic agents have perfect information etc. as required by *REH*; hence, they must rely on other processes such as bounded rationality (Sargent, 1993). Such a criticism would be an *external criticism,* i.e., the assumptions cannot obtain in reality, hence we must drop *REH* (but if the assumptions obtained, *REH* would be all right). The criticism coming from chaos theory is an *internal* criticism, all the more destructive because it shows that *REH* is untenable even when its assumptions obtain.

In fact, if the true model is chaotic, economic agents - assumed to have perfect information including knowledge of the model (exactly as a physicist knows the equations governing a certain phenomenon) - cannot conceivably achieve the infinite precision required to avoid the devastating effects of sensitive dependence on initial conditions.

Perfect deterministic foresight *out of steady states* (Grandmont, 1985, Sect. 3) would be impossible in economics as it is in physics when the model is chaotic. The situation would become worse if, in addition to chaos, stochastic elements were also present.

Even if economic agents knew the stochastic process driving the exogenous shocks, the presence of SDIC on the deterministic part would make it impossible for them to calculate the objective probability distribution of outcomes.

Hence, it would not be possible to verify the essential properties (see Gandolfo, 2010, Sect. 28.4.1) of rational expectations in a stochastic context. In both cases (deterministic and stochastic), the economic theorist would have to abandon the assumption of rational expectations and rely on other rules for expectation formation. Heiner (1989) has suggested a form of adaptive expectations. From the theoretical point of view, rules for expectation formation *should be consistent with the underlying chaotic model,* but *general* rules of this type have not yet been devised (Medio, 1992, p. 18). In the meantime, rules of thumb used by practical agents might have to be taken into account, with the proviso that these rules have a sense only in the very short run.

A second implication of chaos concerns the use of econometric models in forecasting (Baumol and Quandt, 1985). Estimated parameters in econometric models have a confidence interval - which means that the “true” value may be anywhere within this interval with the assumed probability. But even if this confidence interval could be shrunk almost to zero(which is practically impossible), it takes no econometrician to understand that - if the “true” model is chaotic - the presence of *SDIC* implies the impossibility of forecasting except for maybe the very short run (on short-run predictability in chaotic models see also H.-W. Lorenz, 1993, Chap. 6, Sect. 6.4).

A third implication of chaos is the *irreversibility of time* ***in theory***. This can easily be seen by considering unimodal maps (see Gandolfo, 2010, Sect. 25.2.1). As pointed out by Barnett and Chen(1988, p. 203), the existence of a turning point in the map f makes f non-invertible because the inverse of f is set-valued. While f is a function, the inverse of f is a correspondence. This means that, while “normal” equations can be integrated, in principle, either forward or backward in time, only forward integration is possible here.

We would like again to stress that this is time irreversibility *in theory,* unlike time irreversibility "in practice", which occurs in "normal" dissipative systems that can be in principle integrated either forward or backward in time but in practice do not allow a correct "prediction" of the past (on this point see Gandolfo, 2010, Sect. 23.4.2).

This as regards the theory. In physics and related disciplines, apart from purely theoretical interest, important results in the study and explanation of real phenomena have been obtained. Hence, the economist with a more applied bent would undoubtedly ask to be shown that chaotic models give *better* explanations of *real* economic phenomena than *non-linear non-chaotic stochastic* models.

Let us stress that we are *not* suggesting to compare chaotic models (which are necessarily non-linear) with linear stochastic models, but with non-linear dynamic models that are non-chaotic but stochastic.

However, the distinction between *endogenous* a periodic behavior (coming from a non-stochastic chaotic model) and *exogenous* a periodic behavior (coming from a non-chaotic model with exogenous stochastic disturbances) is slippery in applied work, since - generally speaking - it depends on the size of the model. A small model takes all the rest as exogenous, and it might be unwarranted to assume that the exogenous rest has no influence on the endogenous variables. Since there are a large number of exogenous factors at work, to a first approximation it may be reasonable to assume a stochastic influence (which is of course the standard justification for adding a stochastic disturbance term in econometric models).

When the small model is considered as part of a larger model, some of the exogenous influences may be taken in as endogenous variables. This has induced some to argue that the distinction is not only slippery, but also meaningless: "Whether fluctuations are endogenously or exogenously generated, stochastic or deterministic, is a property of a model, not of the real world. Only if there were a true model in much more precise correspondence with the real world than are macroeconomic models might it be a useful shorthand to speak of the actual business cycle as being stochastic or deterministic" (Sims, 1994, p. 1886).

This shows that the main problem in applied economics is one of empirical *detection*, namely whether actual economic data show *evidence of chaos as distinct from the behavior deriving from a non-linear non-chaotic stochastic system*. Various tests have been developed (the basic references are Brock et al., 1991; Pesaran and Potter eds., 1992; Barnett and Binner eds., 2004, Part 4), but it should be emphasized that it is not enough to show the presence of non-linearity in the data.

Non-linearity is a necessary, but by no means a sufficient condition for chaos, and can at best show that *linear* stochastic models are not suitable: it cannot discriminate between non-linear stochastic and non-linear deterministic random behavior, which is what we are concerned with.

Specific tests to discriminate between chaotic systems and non-linear non-chaotic stochastic systems, such as the correlation dimension and the maximum Liapunov exponent[7], do exist, but they do not work very well in uncovering low-dimensional chaos when stochastic noise is present (Mirowski,1990; Liu et al., 1992; Chen, 1993; Granger, 1994; Barnett and Serletis, 2000; Hommes and Manzan, 2006). Real economic data usually show non-linearity but generally fail to exhibit low-dimensional chaos (Liu et al., 1992; Day, 1994; Granger, 1994; Serletis and Shintani, 2006; Shintani, 2008). This is the case also when one has enough observations (one of the problems with these tests is that they require several thousand observations to be reliable) as when testing chaos on daily exchange rates (DeGrauwe et al., 1993). Federici and Gandolfo (2002) found no evidence of chaos in the series of the lira/$ daily exchange rate (5417 observations). Andrangi et al. (2010), employing the daily bilateral exchange rates of the U.S. dollar against the Canadian dollar, yen, and Swiss franc, conducted a battery of tests for the presence of chaos.

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[7]A necessary but not sufficient condition in order to define a system as being chaotic is that the strange attractor has a fractal dimension. In the literature, there are many methods for calculating the fractal dimension (Hausdorff dimension, the box-counting dimension, the information dimension, and the correlation dimension). Among these different algorithms, the correlation dimension proposed by Grassberger-Procaccia (1983) and based on phase space reconstructions of the process to estimates, has the advantage of being straightforward and quickly implemented. The correlation dimension can only distinguish low-dimensional chaos from high-dimensional stochastic processes, particularly with economic data. The Liapunov exponent is a measure of the rate at which nearby trajectories in phase space diverge (thus, it is a measure of *SDIC*). Chaotic orbits have at least one positive Liapunov exponent. For periodic orbits, all Liapunov exponents are negative. The Liapunov exponent is zero near a bifurcation. In general, there are as many exponents as there are dynamical equations.

While they find strong evidence of nonlinear dependence in the data, the evidence is not consistent with chaos[8].

To sum up, economists have followed various approaches to examine the question of chaos in economic variables.

I) On the one hand, there are studies that simply examine the data and apply various tests, such as those mentioned above. These tests have been originally developed in the physics literature.

II) On the other hand, structural models are built and analyzed. This analysis can be carried out, in principle, in several ways:

II.a) showing that plausible economic assumptions give rise to theoretical models having dynamic structures that fall into one of the mathematical forms known to give rise to chaotic motion;

II.b) building a theoretical model and then

II.b_1) giving plausible values to the parameters, simulating the model, and testing the resulting data series for chaos; or

II.b_2) estimating the parameters econometrically, and then proceeding as in b_1.

Approach I) is not very satisfactory from our point of view. Actually, once one has detected the presence of chaos in the data, the next question is: so what? In fact, we believe that the correct approach is to try to find the dynamic model (if any) underlying the data. This is essential for explaining the phenomenon under examination as well as for possible control of the chaotic motion (on chaos control see, for example, Gandolfo, 2010, Sect. 25.5). Besides, in the case of the investigation of individual time series to determine whether they are the result of chaotic or stochastic behavior, the results could be inconclusive, as shown in the single blind comparative study of Barnett et al. (1997).

Approach II.a) is more interesting, but has no connection with the data. This connection is present in approach II.b), and in particular in approach II. b_2), that we deem the most satisfactory.

[8]Barnett et al. (see Barnett and Binner eds., 2004, Chap. 26) have run an interesting competition between various tests for chaos, with different results from different tests.

3. Chaos in Exchange Rates

The various ways to approach chaos in economics can be well exemplified by considering exchange rates.

After the failure of the standard structural models of exchange rate determination in out-of-sample ex-post forecasts, (the most notable empirical rejection is that by Meese and Rogoff, 1983a,b; for subsequent studies see Gandolfo et al., 1990, 1993; Gandolfo, 2002, and Rogoff and Stavrakeva, 2008), the exchange rate has come to be considered as a stochastic phenomenon; and exchange rate forecasting has come to rely on technical analysis and time series procedures, with no place for economic theory. Economic theory can be reintroduced in various ways, one of which is through a chaotic model. In fact, this would explain the apparently erratic behavior of the exchange rate not through purely stochastic processes, but as due to a deterministic economic model capable of generating chaos. Another possibility would be to use a non-linear non-chaotic but stochastic structural model.

Furthermore, it has become evident that it is not possible to understand exchange rate behavior by relying on models with representative agents. All forms of this simplifying approach have failed empirically (see Sarno and Taylor, 2002). There is now abundant evidence that market participants have quite heterogeneous beliefs on future exchange rates. These different expectations introduce non-linear features in the dynamics of the exchange rate. Heterogeneous agent models may create complex endogenous dynamics, including chaotic dynamics. This approach was initiated by Frankel and Froot (1987, 1990a, b). Further studies developed this line of research mainly in the context of stock markets (e.g. Kirman, 1991, Day and Huang, 1990; Brock and Hommes, 1997, 1998; Lux, 1998; Le Baron et al. 1999; Gaunersdorfer et al. 2003)[9].

The empirical evidence in favor of chaos in the exchange rate is not very strong. Sometimes chaos has been detected in the data (see Bajo-Rubio et al. 1992; De Grauwe et al. 1993; Chen, 1999; Bask, 2002; Brzozowska-Rup and Orlowski, 2004; Weston, 2007; Torkamani et al., 2007; Das and Das, 2007; Mishra, 2011), but most often no such dynamics has been found (Brooks, 1998; Guillaume, 2000; Federici and Gandolfo, 2002; Serletis and Shahmoradi, 2004; Vandrovych, 2005; Resende and Zeidan, 2008; Adrangi et al., 2010). In general, the empirical evidence for chaotic dynamics in economic time series is very fragile.

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[9]For surveys of this field of literature, see Hommes (2006), Chiarella et al. (2009), Hommes and Wagener (2009), Lux (2009) and Westerhoff (2009).

Existing chaotic exchange rate models (De Grauwe and Versanten, 1990; Reszat, 1992; De Grauwe and Dewachter, 1993a,b; De Grauwe, Dewachter, Embrechts, 1993; De Grauwe and Grimaldi, 2006a,b; Ellis, 1994; Szpiro, 1994; Chen, 1999: Da Silva, 2000, 2001; Moosa, 2000, Chap. 9) follow approaches ($\mathrm{II}.a$) or ($\mathrm{II}.b_1$).

From the theoretical point of view, these models show that with orthodox assumptions (*PPP*, interest parity, etc.) and introducing nonlinearities in the dynamic equations, it is possible to obtain a dynamic system capable of giving rise to chaotic motion. However, none of these models is estimated, and the conclusions are based on simulations: the empirical validity of these models is not tested.

In the present paper, after a preliminary investigation of the data according to I), we show an example of approach $\mathrm{II}.a$) and approach $\mathrm{II}.b_2$).

3.1 **Analysis of the data**

Modern financial economics deals with high frequency, or tick-by-tick financial data. This improves data availability for nonlinear model testing and in particular, for chaos tests. Many researchers believe that understanding the nature of tick-by-tick data is the key to explain the behavior of the exchange rate market. In the first step, we use tick-by-tick Euro/Dollar exchange rate from January 2003 to December 2009 (one-minute and five-minute intervals) to extract evidence of chaotic dynamics. Similarly, to many other papers, we study the exchange rate returns (the exchange rate return at time t is calculated as the log difference of two consecutive exchange rate levels). Tools from dynamical systems theory, such as the maximum Liapunov exponent, are used.

In addition, we apply the reshuffled (surrogate) data procedure, which is unfortunately overlooked in most tests carried out in economic studies. The results of this analysis indicate that the data do not possess the features that are required to classify them as chaotic[10].

[10]In addition, artificial Neural Networks, Genetic Programming and Genetic Algorithms in particular, are widely applied to capture nonlinear relations and trends for in forex market (Mendes et al., 2012; Ravi et al., 2012).

4. A Non-Linear Model

Given that non-linearity is a necessary, though not sufficient condition for the onset of chaos, the following problem arises. When one abandons linearity (and related functional forms that can be reduced to linearity by a simple transformation of variables, such as log-linear equations), in general it is not clear which non-linear form one should adopt.

Further, to clarify the matter, let us distinguish between *purely qualitative* non-linearity and *specific* non-linearity.

By *purely qualitative* non-linearity, we mean the situation in which we only know that a generic non-linear functional relation exists with certain *qualitative* properties, such as continuous first-order partial derivatives with a given sign and perhaps certain bounds.

By *specific*non-linearity, we mean the situation in which we assume a specific non-linear functional relationship. Since in general it is not clear from the theoretical point of view which non-linear form one should adopt, the choice of a form is often arbitrary or made for convenience, based on *ad hoc* assumptions.

In our case, however, it is possible to introduce a non-linearity on sound economic grounds. This concerns the excess demand of non-speculators. To understand this point, a brief description of the model (see Federici and Gandolfo, 2012) is called for.

4.1 The model

Our starting point is that the exchange rate is determined in the foreign exchange market through the demand for and supply of foreign exchange.

This is a truism. It should be complemented by the observation that, when all the sources of demand and supply - including the monetary authorities through their reaction function - are accounted for, that is, once one has specified behavioral equations for *all* the items included in the balance of payments, the exchange rate comes out of the solution of an implicit dynamical equation.

Let us then come to the formulation of the excess demands (demand minus supply) of the various agents. Our classification is functional. It follows that a commercial trader who wants to profit from the leads and lags of trade (namely, is anticipating payments for imports and/or delaying the collection of receipts from exports in the expectation of a depreciation of the domestic currency) is behaving like a speculator.

1) In the foreign exchange market non-speculators (commercial traders, etc.) are permanently present, whose excess demand only depends on the current exchange rate.
2) Let us now introduce speculators, who demand and supply foreign exchange in the expectation of a change in the exchange rate. According to a standard distinction, we consider two categories of speculators, fundamentalists and chartists[11].

2a) Fundamentalists hold regressive expectations, namely they think that the current exchange rate will move toward its "equilibrium" value. There are several ways to define such a value[12]; we believe that the most appropriate one is the NATREX (acronym of NATural Real EX change rate), set forth by Stein (1990, 1995, 2001, 2002, 2006). It is based on a specific theoretical dynamic stock-flow model to derive the equilibrium real exchange rate. The equilibrium concept reflects the behavior of the fundamental variables behind investment and saving decisions in the absence of cyclical factors, speculative capital movements and movements in international reserves. Two aspects of this approach are particularly worth noting.

The first is that the hypotheses of perfect knowledge and perfect foresight are rejected: rational agents who efficiently use all the available information will base their intertemporal decisions upon a *sub-optimal feedback control* (*SOFC*) rule, which does not require the perfect-knowledge perfect-foresight postulated by the Representative Agent Intertemporally Optimizing Model, but only requires *current* measurements of the variables involved. The second is that expenditure is separated between consumption and investment, which are decided by different agents. The consumption and investment functions are derived according to *SOFC*, through dynamic optimization techniques with feedback control.

[11] For simplicity's sake, we neglect the possibility of switching between the two categories.
[12] Typically, in the literature the *PPP* value is used as a measure of the equilibrium exchange rate.

Thus, the NATREX approach is actually an intertemporal optimizing approach, though based on different optimization rules.

For a treatment of the NATREX, and for an empirical estimation of the $/€ NATREX, see Belloc, Federici and Gandolfo (2008), and Belloc and Federici (2010). The nominal NATREX is exogenously given and assumed known by fundamentalists.

2b) The excess demand by chartists is given by a function of the difference between the expected and the actual exchange rate, where the expected exchange rate is based on extrapolative expectations.

3) Finally, suppose that the monetary authorities are also operating in the foreign exchange market with the aim of influencing the exchange rate[13], account being taken of the NATREX, by using an integral policy *à la* Phillips. They aim either at stabilizing the exchange rate around its NATREX value, or at maintaining or generating a situation of competitiveness.

The mathematics of the various excess demand functions is described in detail in Federici and Gandolfo (2012). By imposing the equilibrium condition in the foreign exchange rate, we arrive to a third-order non-linear differential equation, where the non-linearity is of the purely qualitative type described above. It is however possible to introduce a specific non-linearity on sound economic grounds. This concerns the excess demand for foreign exchange by non-speculators.

To understand this point, a digression is called for on the derivation of the demand and supply schedules of these agents.

4.2 Derivation of the Demand and Supply Schedules of Non-Speculators[14]

The main peculiarity of these demand and supply schedules for foreign exchange is the fact that they are *derived* or *indirect* schedules in the sense that they come from the underlying demand schedules for goods (demand for domestic goods by nonresidents and demand for foreign goods by residents).

[13]Central bank have often used direct interventions as a tool to stabilize short-run trends or to correct long-term misalignments of the exchange rate. The large empirical literature on the impact and the effectiveness of these interventions provides mixed evidence (see Beine et al. 2009; Beine et al. 2007; Dominguez, 2006;Humpage, 2003 among others).

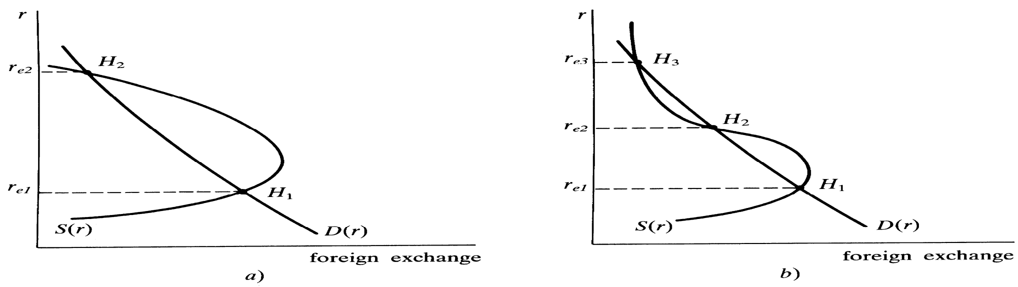
[14]For an in-depth treatment of this point, see Sect. 7.3.1 in Gandolfo, 2002.

In other words, in the context we are considering, transactors do not directly demand and supply foreign exchange as such, but demand and supply it because of the underlying demands for goods. Thus the demand for and supply of foreign exchange depend on the elasticities of the underlying demands for goods. Consider for example $S(r)$ the total revenue of foreign exchange from exports (determined by export demand), which depends on the elasticity of export demand.

If the elasticity of exports is greater than one, an exchange-rate depreciation of, say, one per cent, causes an increase in the volume of exports greater than one per cent, which thus more than offsets the decrease in the foreign currency price of exports: total receipts of foreign exchange therefore increase. The opposite is true when the elasticity is lower than one.

Since a varying elasticity is the norm rather than an exception (a simple linear demand function has a varying elasticity), cases like those depicted in Fig. 1 are quite normal.

Figure1: Non- linear supply function



In the case depicted in Fig. 1a) the function $S(r)$ can be represented by a quadratic, while in the case of Fig. 1b) a cubic might do. Let us consider the simpler quadratic case, $S(r) = a + br + cr^2, a > 0, b > 0, c < 0,$ where $a, b, c,$ are constants[15].

[15]We have chosen the quadratic form for simplicity's sake and because of the parsimony principle. Besides, running a quadratic and a cubic interpolation on the data for $S(r)$ and r did not give substantially different results.The quadratic function $a + br + cr^2$ as represented in the diagram implies $a > 0, b > 0, c < 0.$

This nonlinearity creates the possibility that a depreciation of the currency does not improve the trade balance (the Marshall-Lerner condition is not satisfied), leading to an increasing net demand for foreign exchange[16]. For example, Edwards (2005), Chinn (2007), Belloc and Federici (2010) find that the relation between the US exchange rate movements and the behavior of the US current account does not satisfy the Marshall-Lerner condition.

By using the quadratic approximation examined above, we arrive at a jerk differential equation[17], which is known to give, possibly, rise to chaos for certain values of the parameters [Sprott, 1997, eq. (8)]. This completes approach $\mathrm{II}.a$).

4. Empirical Results

The final step, which brings us to approach $\mathrm{II}.b_2$), is to econometrically estimate the parameters of the jerk differential equation by using appropriate techniques. Estimates of the parameters were found by a Gaussian estimator of the non-linear model, subject to all constraints inherent in the model, by using Wymer's software for the estimation of continuous time non-linear dynamic models (see Wymer, various dates).

We use daily observations of the nominal Euro/Dollar exchange rate over the period January 2, 1975 to December 29, 2003 (weekends and holidays are neglected)[18].

The derivation of the NATREX series is discussed in detail in Federici and Belloc (2010); in that paper the NATREX is also compared with the observed €/$ market rate both by diagrams and by calculating a misalignment index[19]

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16 This feature might disappear in the long run, but the investigation of this possible outcome lies outside the scope of the present paper.

17 A jerk function has the general form

$$x''' = F(x'', x', x)$$

In physical terms, the jerk is the time derivative of the acceleration.

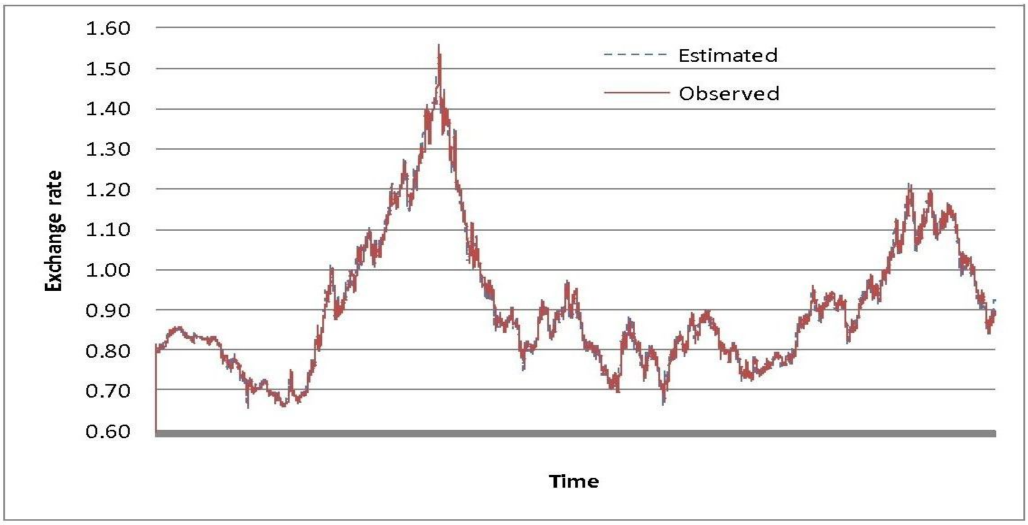
It seems that the denomination jerk came to the mind of a physics student traveling in a car of the New York subway some twenty years ago. When standing in a subway car it is easy to balance a slowly changing acceleration. However, the subway drivers had a habit of accelerating erratically (possibly induced by the rudimentary controls then in use). The effect of this was to generate an extremely high jerk.

18 Source: EUROSTAT.

19 A feature of the continuous time methodology is that, once a model has been estimated with the available data (e.g., quarterly data), it is possible to generate observations at any time frequency (see, for

The estimation of the model shows a remarkable agreement between estimates and theoretical assumptions. In fact, not only all the coefficients have the expected sign and are highly significant, but in addition, the observed and the estimated values are very close, as shown by Fig. 2.

Figure2: Observed and estimated values



The in-sample root mean square error (*RMSE*) of forecasts[20] of the endogenous variable r turns out to be 0.005475, a very good result.

As regards the out-of-sample, ex post forecasts, we simulated the model over the period January 5, 2004 to June 30, 2006 (weekdays only) and obtained a *RMSE* of 0.091338. This value, although higher than the in-sample value (which is a normal occurrence), is satisfactory.

example, Gandolfo, 2009, Chap. 26 Sect. 26.2; Wymer, 1997). Thus, we could generate daily observations of the NATREX.

[20]To obtain these forecasts, the differential equation is re-initialized and solved n times (if one wants forecasts for n periods), each time using the observed value of the endogenous variable in period t as initial value in the solution, which is then employed to obtain the forecast for period $t+1$. In other words, the re-initialization is at the same frequency as the sample observations.

After the studies of Meese and Rogoff (1983a,b), it has become customary to evaluate the forecasting performance of an exchange rate model by comparing it with that of a random walk[21]. In our case we have in-sample *RMSE*=0.01089 (almost twice as that of our model), and out-of-sample *RMSE*=0.10238. This is slightly worse than that of our model.

4.1 Testing for chaos

Our first step[22] was that of looking for a strange attractor through *phase diagrams*.

Figure3: Phase diagram

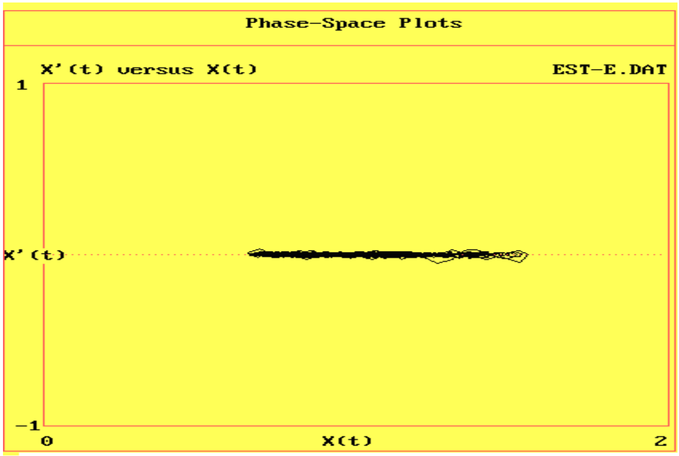


Fig. 3 plots $r'(t)$ against $r(t)$ (these are denoted by $X'(t), X(t)$ in the figure). No discernible structure appears. There does not seem to be a point around which the series evolves, approaching it and going away from it an infinite number of times.

On the contrary, the values are very close and no unequivocal closed orbits or periodic motions seem to exist. If we lengthen the time interval for which the phase diagram is built we obtain closed figures, but we cannot clearly classify them as strange attractors because when the data contain such an attractor, this should remain substantially similar as the time interval changes.

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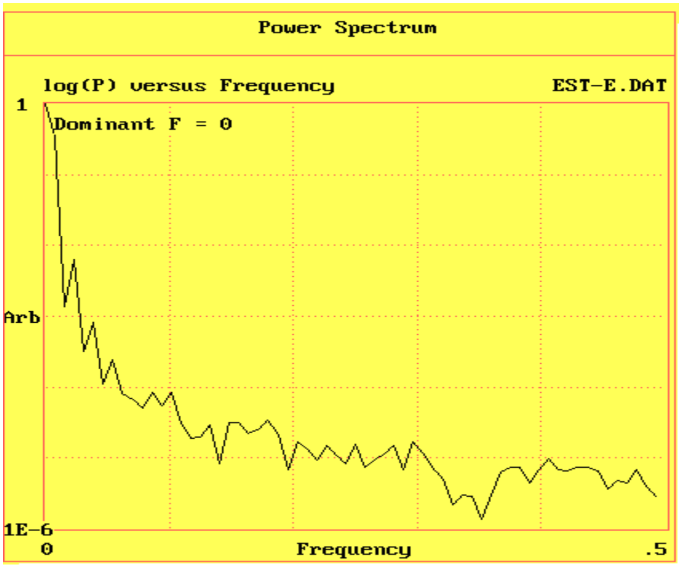
[21]This type of evaluation, once standard, is considered doubtful in the recent literature: see, e.g., Rogoff and Stavrakeva (2008), Lam et al (2008). However, this debate lies outside the scope of the present paper.

[22]The tests were carried out using the software Chaos Data Analyzer by Sprott and Rowlands (1992).

Such a feature is absent. This test, however, is hardly conclusive, as it relies on impression rather than on quantitative evaluation.

We then computed the *power spectrum* (Fig. 4). Power spectra, that are straight lines on a log-linear scale, are thought to be good candidates for chaos. This is clearly not the case.

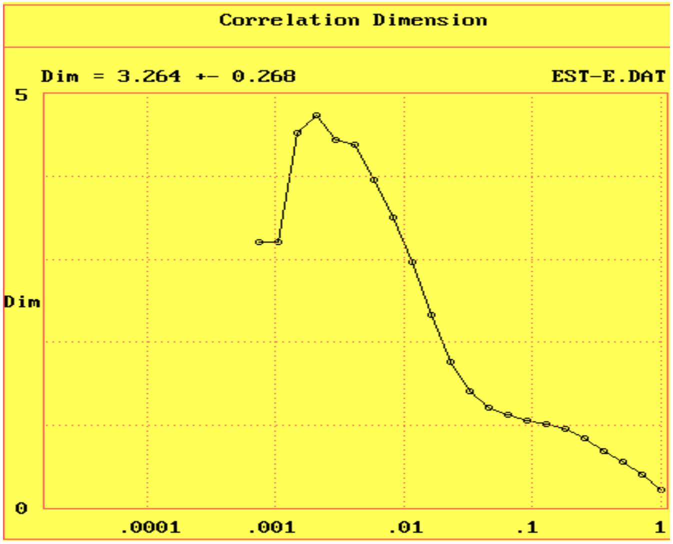
Figure4: Power spectrum



Quantitative tests are based on the *correlation dimension* and *Liapunov exponents*.

The Grassberger-Procaccia algorithm for the computation of the *correlation dimension* requires the presence of a flat plateau in the diagram where the log of the dimension is plotted against the log of the radius.

Figure5: Correlation dimension



Since no such plateau exists (see Fig. 5), the computation of the dimension (which turned out to be 3.264 ± 0.268) is not reliable. In any case, it should be noted that saturation of the correlation dimension estimate is just a necessary, but not sufficient, condition for the existence of a chaotic attractor, since also non-linear non-chaotic stochastic systems are capable of exhibiting this property (Scheinkman and LeBaron, 1989; for explicit examples of stochastic processes with finite correlation dimensions see Diks, 2004).

Arguably, Liapunov exponents provide the only test specific for chaos. The Liapunov exponent is a measure of the rate at which nearby trajectories in phase space diverge (*SDIC*). Chaotic orbits have at least one positive Liapunov exponent.

Inserting the estimated parameters into the original non-linear model and solving the differential equation, we obtained the values (daily data) of the exchange rate generated by the model. Then we applied to this series the Liapunov exponents test. In this case the greatest Liapunov exponent is 0.103 ± 0.016 . This is evidence for chaos, but the reshuffled (surrogate) data procedure refutes such a result.

The basic idea is to produce from the original data a new series with the same distributional properties but with any non-linear dependence removed. The maximum Liapunov exponent test is then applied to this surrogate series to check whether it gives the same (pro chaos) results as those obtained from the original series.

If the results are the same, we should suspect the veracity of our conclusions. We obtained a positive largest Liapunov exponent, 0.419 ± 0.16.

Hence, we can conclude that the series generated by the estimated model cannot be considered as chaotic.

The previous results are confirmed by a different procedure, which is the following.

Liapunov exponents have been calculated from the underlying non-linear model[23] for the estimated parameter values, using the variational matrix equation, and these concentrate information on the nature of the non-linear dynamics. In our case, all exponents are negative, and are $-0.218691E^{-3}$, $-0.620225E^{+01}$, $-0.620229E^{+01}$. Based on these results the model is stable dynamically and structurally stable (i.e. the results did not change in a substantial way even for large changes in the parameter values. Thus, the estimated parameters are not in the chaotic region, but one might suspect that they are close to the first bifurcation (so close to instability)[24]. It would be possible to check for bifurcations by considering the linear approximating system around a reference point or path (for example the steady state)[25], but such an analysis would be valid only in a small neighborhood of the reference solution. Rather, we prefer to ascertain what happens for sufficiently wide variations in the parameters.
To start with, we have found that if parameter a_2 is set to zero, which means that fundamentalists are not active in the market, the model is unstable[26]. Moreover, in a fairly wide neighborhood of the other parameters, the model remains unstable.

There is a major change in the dynamic structure depending on whether or not fundamentalists are in the market. However, $a_2 = 0$ is a rather drastic assumption, and conflicts with our empirical results, according to which a_2 is significantly different from zero.

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[23] See Wymer (2009) on the advantage of calculating Liapunov exponents from an estimated model.
[24] For example, the Brock and Hommes model (1998) can generate chaos, but using the estimated parameters in Boswijk et al. (2007) no chaos arises. However, these parameters are close to the first bifurcation.
[25] For example by using, the procedure set forth by Barnett and He, 1998.
[26] One Liapunov exponent was positive, and the other two were negative and different.

A more reasonable approach seems to be the following. Since we are dealing with *estimated* parameters, it follows that the true value of the parameter can lie anywhere in the confidence interval at the given probability level (on this point see Gandolfo, 1992).

At the 5%, level the confidence interval is calculated as point estimate $\pm 1.96\sigma,$ where σ is the *ASE*; at the 1% level, it is point estimate $\pm 2.58\sigma.$ Thus, we simulated the model with $a_2 = 11.443$ (lower bound of the 5% interval) and with $a_2 = 9.693$ (lower bound of the 1% interval). With both values, no chaos appeared (the Liapunov exponents remained all negative)[27].

5. Conclusion

Our results have important economic implications

I) The implications for the foreign exchange market, and almost certainly other financial markets, are striking. The stabilizing role of fundamentalists is not surprising given their longer horizons, but the need for fundamentalists to stabilize a market that would otherwise be unstable raises questions about the role of the other players. In recent years, it has been argued that day-traders and other short-term players are important in providing liquidity to the market. If so, they should make the market more stable but they do not.

II) The second implication is methodological. As stated in the Introduction, after the failure of the standard structural models of exchange rate determination in out-of-sample ex-post forecasts (the most notable empirical rejection was that by Meese and Rogoff, confirmed by subsequent studies), exchange rate forecasting has come to rely on technical analysis and time series procedures, with no place for economic theory. Economic theory can be reintroduced:

a) Through a non-linear purely deterministic structural model giving rise to chaos.
b) Through a non-linear non-chaotic but stochastic structural model.

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[27]The Liapunov exponents remained negative for increasingly smaller positive values of a_2 (well below the confidence interval). These exponents, while negative, were smaller and smaller in absolute value as a_2 decreased toward zero.

The fact that our model fits the data well but does not give evidence for chaos seems to point to the abandonment of *linear* stochastic models in favor of *non-linear* (but non-chaotic) stochastic models rather than in favor of chaotic models.

Hence, until new cogent empirical evidence is presented, we feel that - though chaotic dynamics is an important and welcome addition to the dynamic economist' tool kit - it is a little soon to declare its undisputed pre-eminence in economics. On this, the jury is still out[28].

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[28]Other authors are more drastic: "Chaos models have generated a vast literature and have been very successful in the physical sciences. However they have been much less successful in the social sciences and their day seems to have passed" (Granger, 2008, p. 2).

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